

THE HO-ZHAO PROBLEM

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ABSTRACT. Given a poset P , the set, $\Gamma(P)$, of all Scott closed sets ordered by inclusion forms a complete lattice. A subcategory \mathbf{C} of \mathbf{Pos}_d (the category of posets and Scott-continuous maps) is said to be Γ -faithful if for any posets P and Q in \mathbf{C} , $\Gamma(P) \cong \Gamma(Q)$ implies $P \cong Q$. It is known that the category of all continuous dcpos and the category of bounded complete dcpos are Γ -faithful, while \mathbf{Pos}_d is not. Ho & Zhao (2009) asked whether the category \mathbf{DCPO} of dcpos is Γ -faithful. In this paper, we answer this question in the negative by exhibiting a counterexample. To achieve this, we introduce a new subcategory of dcpos which is Γ -faithful. This subcategory subsumes all currently known Γ -faithful subcategories. With this new concept in mind, we construct the desired counterexample which relies heavily on Johnstone's famous dcpo which is not sober in its Scott topology.

1. INTRODUCTION

The collection, $\Gamma(X)$, of closed subsets of a topological space X , ordered by inclusion, forms a distributive complete lattice often referred to as the *closed set lattice* of X . If a lattice L is isomorphic to the closed set lattice of some topological space X , we say that X is a *topological representation* of L . It is natural to ask:

Question 1. Which lattices have topological representations?

Seymour Papert was the first to characterize such lattices as those which are complete, distributive, and have a base consisting of irreducible elements, [9].

We can also ask how much of the topological structure of a space is encoded in its closed set lattice. Following Wolfgang Thron, [11], we say that two topological spaces X and Y are

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lattice-equivalent if their closed-set lattices are order-isomorphic. Clearly, homeomorphic spaces are lattice-equivalent, but the converse fails. This then leads us to:

Question 2. Which classes \mathbf{C} of topological spaces are such that any two lattice-equivalent spaces X and $Y \in \mathbf{C}$ are homeomorphic, i.e.:

$$\forall X, Y \in \mathbf{C}. \Gamma(X) \cong \Gamma(Y) \implies X \cong Y ?$$

Sober topological spaces are exactly those that can be fully reconstructed from their closed set lattices, [3]; therefore the class **Sob** of sober spaces is a natural choice in answer to Question 2. Furthermore, any topological space X is lattice-equivalent to its sobrification X^s and so it follows that \mathbf{C} cannot contain a non-sober space X and its sobrification X^s at the same time, in other words, **Sob** is a *maximal* choice for \mathbf{C} .

The two questions above can also be asked in the context of a particular class of topological spaces. The ones we have in mind were introduced by Dana Scott, [10, 4], and are known collectively as *domains*. The characteristic feature of domains is that they carry a partial order and that their topology is completely determined by the order. More precisely, let P be a poset and U a subset of P . One says that U *Scott open*, if (i) U is an upper set, and (ii) U is inaccessible by directed joins. The set $\sigma(P)$ of all Scott opens of P forms the *Scott topology* on P , and $\Sigma P := (P, \sigma(P))$ is called the *Scott space* of P . In what follows, for a poset P , we write $\Gamma(P)$ to always mean the lattice of Scott-closed subsets of P .

We may now relativize our definitions to the context of Scott spaces. We say that a lattice L has a *Scott-topological representation* if L is isomorphic to $\Gamma(P)$ for some poset P , and ask:

Problem 3. Which lattices have Scott-topological representations?

Although some work has been done on this problem, [6], as of now it remains open. In the special case of *continuous domains* a very pleasing answer was given independently by Jimmie Lawson, [8], and Rudolf-Eberhardt Hoffmann, [5]. They showed that a lattice L has a Scott-topological representation $\Gamma(P)$ for some continuous domain P if and only if L is completely distributive.

In the order-theoretic context the second question reads as follows:

Problem 4. Which classes of posets \mathbf{C} satisfy the condition

$$\forall P, Q \in \mathbf{C}. \Gamma(P) \cong \Gamma(Q) \implies P \cong Q ?$$

A class \mathbf{C} of posets is said to be Γ -*faithful* if the above condition holds ([12][p. 2170, Remark 2]). The following classes of posets are known to be Γ -faithful:

- (1) **CSL** of complete semilattices, [6], containing in particular all complete lattices;
- (2) **SOB** $_{\sigma}$ of dcpos whose Scott topologies are sober, containing in particular **Cont**, the class of all continuous dcpos.

A *dcpo-completion* of a poset P is a dcpo A together with a Scott-continuous mapping $\eta : P \longrightarrow A$ such that for any Scott-continuous mapping $f : P \longrightarrow B$ into a dcpo B there exists a unique Scott-continuous mapping $\hat{f} : A \longrightarrow B$ satisfying $f = \hat{f} \circ \eta$. It was shown in [12] that the dcpo-completion, $E(P)$, of a poset P always exists; furthermore $\Gamma(P) \cong \Gamma(E(P))$. Hence the class, **POS**, of all posets is *not* Γ -faithful. Indeed any class of posets that is strictly larger than **DCPO**, the class of dcpos, is not Γ -faithful. This means that we can restrict attention entirely to dcpos, in particular, it leads one to ask:

Problem 5. ([6], [12, Remark 2]) Is **DCPO** Γ -faithful?

This question was dubbed the *Ho-Zhao problem* in [1]. The authors of this paper claimed that the two dcpos

$$\Upsilon = ([0, 1], \leq) \text{ and } \Psi = (\{[0, a] \mid 0 < a \leq 1\}, \subseteq)$$

show that **DCPO** is not Γ -faithful. However, it is easy to see that $\Psi \cong ((0, 1], \leq)$ so that $\Gamma(\Psi) \cong ([0, 1], \leq)$. On the other hand, $\Gamma(\Upsilon) \cong ([0, 1], \leq)_\perp$ which is not isomorphic to $\Gamma(\Psi)$. This failure is only to be expected as both Υ and Ψ are continuous dcpos and we already noted that **Cont** is Γ -faithful.

This paper comprises two parts. In Section 2 we present a positive result by introducing the class **domDCPO** of *dominated dcpos* and showing it to be Γ -faithful. Importantly, **domDCPO** subsumes all currently known Γ -faithful classes listed above. In the second part (Sections 3–5) we show that the answer to the Ho-Zhao problem is negative. We construct a dcpo \mathcal{H} which is not dominated, and derive from it a dominated dcpo $\widehat{\mathcal{H}}$ so that $\mathcal{H} \not\cong \widehat{\mathcal{H}}$ but $\Gamma(\mathcal{H}) \cong \Gamma(\widehat{\mathcal{H}})$. The construction makes use of Johnstone’s famous example of a dcpo \mathcal{S} whose Scott topology is not sober ([7]). To familiarize the reader with Johnstone’s dcpo \mathcal{S} , we revisit it in Section 3, highlighting its peculiarities. This prepares us for the counterexample \mathcal{H} , presented in Section 4. For quick accessibility we keep the description of \mathcal{H} in Section 4 informal. Intrepid readers who are keen to pursue the detailed construction of \mathcal{H} and how it works in answering the Ho-Zhao problem may then continue their odyssey into Section 5.

For notions from domain theory we refer the reader to [2, 4].

2. A POSITIVE RESULT

2.1. Irreducible sets. A nonempty subset A of a topological space $(X; \tau)$ is called *irreducible* if whenever $A \subseteq B \cup C$ for closed sets B and C , $A \subseteq B$ or $A \subseteq C$ follows. We say that A is *closed irreducible* if it is closed and irreducible. The set of all closed irreducible subsets of X is denoted by \widehat{X} . Two facts about irreducible sets will be important below:

- If A is irreducible then so is its topological closure.
- The direct image of an irreducible set under a continuous function is again irreducible.

In this paper we are exclusively interested in dcpos and the Scott topology on them. When we say “(closed) irreducible” in this context we always mean (closed) irreducible with respect to the Scott topology. It is a fact that every directed set in a dcpo is irreducible in this sense, while a set that is directed and closed is of course a principal ideal, that is, of the form $\downarrow x$. Peter Johnstone discovered in 1981, [7], that a dcpo may have irreducible sets which are not directed, and closed irreducible sets which are not principal ideals; we will discuss his famous example in Section 3 and someone not familiar with it may want to have a look at it before reading on.

Given that we may view irreducible sets as a generalisation of directed sets, the following definition suggests itself:

Definition 2.1. A dcpo $(D; \leq)$ is called *strongly complete* if every irreducible subset of D has a supremum. In this case we also say that D is an *scpo*. A subset of an scpo is called *strongly closed* if it closed under the formation of suprema of irreducible subsets.

Note that despite this terminology, strongly complete partial orders are a long way from being complete in the sense of lattice theory.

2.2. Categorical setting. The basis of Questions 1 and 2 in the Introduction is the well-known (dual) adjunction between topological spaces and frames, which for our purposes is expressed more appropriately as a (dual) adjunction between topological spaces and coframes:

$$\Gamma : \mathbf{Top} \overset{\perp}{\rightleftarrows} \mathbf{coFrm}^{\text{op}} : \text{spec}$$

Here spec is the functor that assigns to a coframe its set of *irreducible* elements, that is, those elements a for which $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$,¹ topologized by closed sets $B_b = \{a \in \text{spec}(L) \mid a \leq b\}$. Starting with a topological space $(X; \tau)$ we obtain the *sobrification* X^s of X by composing spec with Γ . Concretely, the points of X^s are given by the closed irreducible subsets of X and the topology is given by closed sets $B^s = \{A \in X^s \mid A \subseteq B\}$ where $B \in \Gamma(X)$.

In order to present Problems 3 and 4 in a similar fashion, it seems natural to replace **Top** with the category of *dcpo spaces*, i.e., directed-complete partially ordered sets equipped with the Scott topology. However, while we know that spec yields topological spaces which are dcpo's in their specialisation order, it is *not the case* that the topology on $\text{spec}(L)$ equals the Scott topology with respect to that order; all we know is that every B^s is Scott-closed.

Rather than follow a topological route, therefore, we reduce the picture entirely to one concerning *ordered sets*. To this end we restrict the adjunction above to the category **MCS** of *monotone convergence spaces* ([4, Definition II-3.12]) on the topological side and compose it with the adjunction between monotone convergence spaces and dcpo's

$$\Sigma : \mathbf{DCPO} \overset{\perp}{\rightleftarrows} \mathbf{MCS} : \text{so}$$

which assigns to a dcpo its Scott space and to a space its set of points with the *specialisation order*. We obtain a functor from **DCPO** to **coFrm** which assigns to a dcpo its coframe of Scott-closed subsets and we re-use the symbol Γ for it rather than writing $\Gamma \circ \Sigma$. In the other direction, we assign to a coframe the *ordered set* of irreducible elements, where the order is inherited from the coframe. To emphasize the shift in perspective, we denote it with irr rather than spec or $\text{so} \circ \text{spec}$. It will also prove worthwhile to recall the action of irr on morphisms: If $h : L \rightarrow M$ is a coframe homomorphism, then $\text{irr}(h)$ maps an irreducible element a of M to $\bigwedge \{x \in L \mid h(x) \geq a\}$. Altogether we obtain the following (dual) adjunction:

$$\Gamma : \mathbf{DCPO} \overset{\perp}{\rightleftarrows} \mathbf{coFrm}^{\text{op}} : \text{irr}$$

Its unit η_D maps an element x of a dcpo D to the principal downset $\downarrow x$, which is always closed and irreducible and hence an element of $\text{irr}(\Gamma(D))$. The counit ε_L (as a concrete map between coframes) sends an element b of a coframe L to the set $\{a \in \text{irr}(L) \mid a \leq b\}$ which is clearly closed under directed suprema, hence an element of $\Gamma(\text{irr}(L))$.

Combining irr with Γ yields a monad on **DCPO** which we denote with $\widehat{(\)}$. Concretely, it assigns to a dcpo D the set \widehat{D} of closed irreducible subsets ordered by inclusion. We call this structure the *order sobrification* of D . If $f : D \rightarrow E$ is a Scott-continuous function between dcpo's, and $A \subseteq D$ is a closed irreducible set, then $\widehat{f}(A)$ is the Scott closure of the direct image $f(A)$ (which is again irreducible as we noted at the beginning of this

¹Strictly speaking, this is the definition for a *coprime* element but in distributive lattices there is no difference between the two.

section). The monad unit is given by the unit of the adjunction and the multiplication $\mu_D = \text{irr}(\epsilon_{\Gamma D}): \widehat{\widehat{D}} \rightarrow \widehat{D}$ maps a closed irreducible collection² of closed irreducible subsets to their union. We will see in Section 4 that this order-theoretic monad is *not* idempotent; this is an important difference to its topological counterpart, the sobrification monad.

The concrete description of μ as union allows us to conclude immediately:

Proposition 2.2. For any dcpo D , \widehat{D} is strongly complete.

There is more to discover about our adjunction and the union operation:

Proposition 2.3. Let D be a dcpo.

- (1) If B a closed set of D then $\varepsilon_{\Gamma(D)}(B) = \{A \in \widehat{D} \mid A \subseteq B\} \in \Gamma(\widehat{D})$.
- (2) If \mathcal{A} is irreducible as a subset of \widehat{D} then $\bigcup \mathcal{A}$ is irreducible as a subset of D .
- (3) If \mathcal{B} is Scott-closed as a subset of \widehat{D} then $\bigcup \mathcal{B}$ is Scott-closed as a subset of D .
- (4) If B is a closed set of D then $\varepsilon_{\Gamma(D)}(B)$ is strongly closed.

Proof. (1) This follows from the general description of the counit.

(2) Assume we have $\bigcup \mathcal{A} \subseteq B \cup C$ with B, C Scott-closed subsets of D . Then $\mathcal{A} \subseteq \varepsilon_{\Gamma(D)}(B) \cup \varepsilon_{\Gamma(D)}(C)$ by (1) and because every member of \mathcal{A} is irreducible. Furthermore, $\varepsilon_{\Gamma(D)}(B)$ and $\varepsilon_{\Gamma(D)}(C)$ are Scott-closed subsets of \widehat{D} , so by assumption, one of them must cover the irreducible collection \mathcal{A} , say $\varepsilon_{\Gamma(D)}(B)$. It then follows that $\bigcup \mathcal{A}$ is contained in B .

(3) This follows for categorical reasons because the union operation is the same as $\Gamma(\eta_D): \Gamma(\widehat{D}) \rightarrow \Gamma(D)$, but a direct proof is instructive for what follows: If S is a directed subset of $\bigcup \mathcal{A}$ then each $x \in S$ must belong to some $A_x \in \mathcal{A}$. Because each A_x is a lower set of D we have $\downarrow x \subseteq A_x$ and because \mathcal{A} is a lower set of \widehat{D} it follows that $\downarrow x \in \mathcal{A}$. The collection $(\downarrow x)_{x \in S}$ is directed and by assumption its supremum $\downarrow \bigvee^\uparrow S$ belongs to \mathcal{A} as \mathcal{A} is Scott-closed. It follows that $\bigvee^\uparrow S$ belongs to $\bigcup \mathcal{A}$.

(4) Let \mathcal{A} be an irreducible subset of $\varepsilon_{\Gamma(D)}(B)$. Then by (2) we have that $\bigcup \mathcal{A}$ is irreducible. Since $\bigcup \mathcal{A}$ is contained in the closed set B , so is its closure A . We know that A is again irreducible and therefore this is the supremum of \mathcal{A} and it belongs to $\varepsilon_{\Gamma(D)}(B)$. \square

Strong closedness is crucial for the following result:

Lemma 2.4. Let D be a dcpo and $\mathcal{B} \in \Gamma(\widehat{D})$.

- (1) $\mathcal{B} \subseteq \varepsilon_{\Gamma(D)}(\bigcup \mathcal{B})$.
- (2) If \mathcal{B} is strongly closed then equality holds.

Proof. The first statement is trivial by the definition of the counit ε . For the second, let A be a closed irreducible subset of $\bigcup \mathcal{B}$. We need to show that A is an element of \mathcal{B} . Every element x of A belongs to some $A_x \in \mathcal{B}$. As we argued in part (3) of the preceding proof, it follows that for every $x \in A$, $\downarrow x \in \mathcal{B}$. We claim that the collection $\mathcal{A} = \{\downarrow x \mid x \in A\}$ is irreducible as a subset of \widehat{D} . This will finish our proof as we clearly have that A is the supremum of \mathcal{A} and by assumption, \mathcal{B} is closed under forming suprema of irreducible subsets.

So let \mathcal{A} be covered by two closed collections $\mathcal{M}, \mathcal{N} \in \Gamma(\widehat{D})$, in other words, every $\downarrow x$ belongs to either \mathcal{M} or \mathcal{N} . It follows that each $x \in A$ belongs to either $\bigcup \mathcal{M}$ or $\bigcup \mathcal{N}$ and these two sets are Scott-closed by part (3) of the preceding proposition. Because A is

²To help the reader we will usually use the words “collection” or “family” rather than “set” when referring to subsets of \widehat{D} .

irreducible, it is already covered by one of the two, and this implies that \mathcal{A} is covered by either \mathcal{M} or \mathcal{N} . \square

2.3. Question 4 revisited. We approach Question 4 via the monad $\widehat{(\)}$. Starting from the assumption $\Gamma(D) \cong \Gamma(E)$ we immediately infer $\widehat{D} = \text{irr}(\Gamma(D)) \cong \text{irr}(\Gamma(E)) = \widehat{E}$ and the question then becomes whether this isomorphism implies $D \cong E$. Our counterexample will demonstrate that in general the answer is “no” but in this section we will exhibit a new class **domDCPO** of *dominated dcpos* for which the answer is positive, that is, we will show:

$$\forall D, E \in \mathbf{domDCPO}. \widehat{D} \cong \widehat{E} \implies D \cong E.$$

Before we do so, let us check that invoking the monad does not change the original question, in other words, the assumption $\widehat{D} \cong \widehat{E}$ is neither stronger nor weaker than $\Gamma(D) \cong \Gamma(E)$:

Proposition 2.5. For arbitrary dcpos D and E , $\widehat{D} \cong \widehat{E} \iff \Gamma(D) \cong \Gamma(E)$.

Proof. The implication from right to left is trivial, so assume we are given an order isomorphism $i: \widehat{D} \rightarrow \widehat{E}$. The idea for an isomorphism ϕ from $\Gamma(D)$ to $\Gamma(E)$ is very simple: Given $B \in \Gamma(D)$ we compute $\varepsilon_{\Gamma(D)}(B)$, the collection of all closed irreducible sets contained in B . Each of these can be replaced with its counterpart in E via the given isomorphism i . In E , then, we simply take the union of the collection $i(\varepsilon_{\Gamma(D)}(B))$. Using the maps that are provided to us by the adjunction, we can express ϕ as follows:

$$\phi: \Gamma(D) \xrightarrow{\varepsilon_{\Gamma(D)}} \Gamma(\widehat{D}) \xrightarrow{\Gamma(i^{-1})} \Gamma(\widehat{E}) \xrightarrow{\cup} \Gamma(E)$$

For an inverse, we follow the same steps, starting at $\Gamma(E)$:

$$\psi: \Gamma(E) \xrightarrow{\varepsilon_{\Gamma(E)}} \Gamma(\widehat{E}) \xrightarrow{\Gamma(i)} \Gamma(\widehat{D}) \xrightarrow{\cup} \Gamma(D)$$

In order to show that these are inverses of each other we use the fact that $\varepsilon_{\Gamma(D)}(B)$ is *strongly* closed which we established in Proposition 2.3(4). Since the concept of strong closure is purely order-theoretic we get that the direct image under i is again strongly closed. This is crucial as it allows us to invoke Lemma 2.4(2). The computation thus reads:

$$\begin{aligned} \psi \circ \phi &= \Gamma(\eta_D) \circ \Gamma(i) \circ \varepsilon_{\Gamma(E)} \circ \bigcup \circ \Gamma(i^{-1}) \circ \varepsilon_{\Gamma(D)} && \text{by definition} \\ &= \Gamma(\eta_D) \circ \Gamma(i) \circ \Gamma(i^{-1}) \circ \varepsilon_{\Gamma(D)} && \text{Lemma 2.4(2)} \\ &= \Gamma(\eta_D) \circ \varepsilon_{\Gamma(D)} && \Gamma \text{ is a functor} \\ &= \text{id}_{\Gamma(D)} && \text{triangle law} \end{aligned}$$

The other composition, $\phi \circ \psi$, simplifies in exactly the same way to the identity on $\Gamma(E)$, and since all maps involved are order-preserving, we have shown that the pair ϕ, ψ constitutes an order isomorphism between $\Gamma(D)$ and $\Gamma(E)$. \square

2.4. Dominated dcpos. Our new version of Question 4 requires us to infer $D \cong E$ from $\widehat{D} \cong \widehat{E}$, and the most direct approach is to find a way to recognize *purely order-theoretically* inside \widehat{D} those elements which correspond closed irreducible subsets of the form $\downarrow x$ with $x \in D$. As our counterexample \mathcal{H} to the Ho-Zhao problem will show, this is not possible for general dcpos. The purpose of the present section is to exhibit a class of dcpos for which the direct approach works.

Definition 2.6. Given $A', A \in \widehat{D}$ we write $A' \triangleleft A$ if there is $x \in A$ such that $A' \subseteq \downarrow x$. We write ∇A for the set $\{A' \in \widehat{D} \mid A' \triangleleft A\}$.

Clearly, an element A of \widehat{D} is of the form $\downarrow x$ if and only if $A \triangleleft A$ holds, but this is not yet useful since the definition of \triangleleft makes explicit reference to the underlying dcpo D . We can, however, record the following useful facts:

Proposition 2.7. Let D be a dcpo and $A \subseteq D$ closed and irreducible.

- (1) $A = \bigvee \nabla A$
- (2) ∇A is irreducible as a subset of \widehat{D} .

Proof. The first statement is trivial because ∇A contains all principal ideals $\downarrow x$, $x \in A$. The proof of the second statement is essentially the same as that of Lemma 2.4. \square

Definition 2.8. Let D be a strongly complete partial order and $x', x \in D$. We write $x' \prec x$ if for all closed irreducible subsets A of D , $x \leq \bigvee A$ implies $x' \in A$. We say that $x \in D$ is \prec -compact if $x \prec x$, and denote the set of \prec -compact elements by $K(L)$.

Note that this is an *intrinsic* definition of a relation on D without reference to any other structure. It is reminiscent of the way-below relation of domain theory but note that it is defined via *closed* irreducible sets. This choice has the following consequence, which is definitely not true for way-below:

Proposition 2.9. Let D be an scpo and $x \in D$. The set $\{a \in D \mid a \prec x\}$ is Scott-closed.

Proof. Let $(a_i)_{i \in I}$ be a directed set of elements, each of which is \prec -below x . We need to show that $\bigvee_{i \in I}^\uparrow a_i$ is also \prec -below x . To this end let A be a closed irreducible subset of D with $x \leq \bigvee A$. Since $a_i \prec x$ for all $i \in I$ we have that every a_i belongs to A and because A is closed, $\bigvee_{i \in I}^\uparrow a_i \in A$ follows. \square

Recall from Proposition 2.2 that \widehat{D} is strongly complete for any dcpo D , so on \widehat{D} we can consider both \triangleleft and \prec . We observe:

Proposition 2.10. $A' \triangleleft A$ implies $A' \prec A$ for all $A, A' \in \widehat{D}$.

Proof. This holds because the supremum of a closed directed collection of closed irreducible subsets is given by union. \square

Our aim now is to give a condition for dcpos D which guarantees the reverse implication.

Definition 2.11. A dcpo D is called *dominated* if for every closed irreducible subset A of D , the collection ∇A is Scott-closed in \widehat{D} .

We are ready for the final technical step in our argument:

Lemma 2.12. A dcpo D is dominated, if and only if $A' \prec A$ implies $A' \triangleleft A$ for all $A, A' \in \widehat{D}$.

Proof. If: Let $A \in \widehat{D}$. By assumption we have $\nabla A = \{A' \in \widehat{D} \mid A' \triangleleft A\} = \{A' \in \widehat{D} \mid A' \prec A\}$ and in Proposition 2.9 we showed that the latter is always Scott-closed.

Only if: Let $A' \prec A$. We know that ∇A is irreducible and by assumption closed. We also know that $A = \bigvee \nabla A$ so it must be the case that $A' \in \nabla A$. \square

We are ready to reap the benefits of our hard work. All of the following are now easy corollaries:

Proposition 2.13. For a dominated dcpo D , the only \prec -compact elements of \widehat{D} are the principal ideals $\downarrow x$, $x \in D$. Also, the unit η_D is an order isomorphism from D to $K(\widehat{D})$.

Theorem 2.14. Let D and E be dominated dcpos. The following are equivalent:

- (1) $D \cong E$.
- (2) $\Gamma(D) \cong \Gamma(E)$.
- (3) $\widehat{D} \cong \widehat{E}$.

Theorem 2.15. The class **domDCPO** of dominated dcpos is Γ -faithful.

2.5. Examples. Let us now explore the reach of our result and exhibit some better known classes of dcpos which are subsumed by **domDCPO**.

Proposition 2.16. Every scpo is dominated.

Proof. If $A' \triangleleft A$ for closed irreducible subsets of an scpo, then by definition $A' \subseteq \downarrow x$ for some $x \in A$ and hence $\bigvee A' \in A$. If $(A_i)_{i \in I}$ is a directed family in ∇A , then $\bigvee_{i \in I}^\uparrow A_i \subseteq \downarrow \bigvee_{i \in I}^\uparrow (\bigvee A_i)$, and the element $\bigvee_{i \in I}^\uparrow (\bigvee A_i)$ belongs to A because A is a Scott-closed set. \square

Corollary 2.17. The following are all dominated:

- (1) complete semilattices and complete lattices;
- (2) dcpos which are sober in their Scott topologies;
- (3) \widehat{D} for any dcpo D .

Proof. (1) holds because complete semilattices have suprema for all bounded sets, which is all that is needed in the proof of the preceding proposition.

(2) holds because the only closed irreducible subsets of sober dcpos are principal ideals.

(3) holds because we showed in Proposition 2.2 that \widehat{D} is always strongly complete. \square

Recall that a topological space is called *coherent* if the intersection of two compact saturated sets is again compact. It is called *well-filtered* if whenever $(K_i)_{i \in I}$ is a filtered collection of compact saturated sets contained in an open O , then some K_i is contained in O already.

Lemma 2.18. Let D be a dcpo which is well-filtered and coherent in its Scott topology. Then for any $A \subseteq D$, the set $\text{ub}(A)$ of upper bounds of A is compact saturated.

Proof. Since $\text{ub}(A) = \bigcap_{a \in A} \uparrow a$, it is saturated. For any finite nonempty $F \subseteq A$, $\bigcap_{a \in F} \uparrow a$ is compact since D is coherent. Thus $\{\bigcap_{a \in F} \uparrow a \mid F \subseteq_{\text{finite}} A\}$ forms a filtered family of compact saturated sets whose intersection equals $\text{ub}(A)$. If it is covered by a collection of open sets then by well-filteredness of D some $\bigcap_{a \in F} \uparrow a$ is covered already, but the latter is compact by coherence so a finite subcollection suffices to cover it. \square

Proposition 2.19. Every dcpo D which is well-filtered and coherent in its Scott topology is dominated.

Proof. Let $(A_i)_{i \in I}$ be a directed family in ∇A . By the preceding proposition, the sets $\text{ub } A_i$ are compact saturated and form a filtered collection. Suppose for the sake of contradiction that $\bigcap_{i \in I} \text{ub}(A_i) \subseteq D \setminus A$. Since $D \setminus E$ is Scott open, by the well-filteredness of D it follows that there exists $i_0 \in I$ such that $\text{ub}(A_{i_0}) \subseteq D \setminus E$. This contradicts the fact that every A_i is bounded by an element of A . \square

At the juncture, the curious reader must be wondering why we have not yet given an example of a dcpo which is not dominated. Now, as we showed, any dcpo whose Scott topology is sober is dominated and so a non-dominated dcpo, if it exists, must be non-sober. Peter Johnstone, [7] was the first to give an example of such a dcpo and it is thus natural to wonder whether it is dominated or not. In order to answer this question we explore his example \mathcal{S} in some detail in the next section. This will also help us to construct and study our counterexample to the Ho-Zhao conjecture.

3. JOHNSTONE'S COUNTEREXAMPLE REVISITED

We denote with $\overline{\mathbb{N}}$ the *ordered set* of natural numbers augmented with a largest element ∞ . When we write \mathbb{N} we mean the natural numbers as a set, that is, *discretely ordered*. Johnstone's counterexample \mathcal{S} , depicted in Figure 1, is based on the ordered set $\mathbb{N} \times \overline{\mathbb{N}}$, that is, a countable collection of infinite chains. We call the chain $C_m := \{m\} \times \overline{\mathbb{N}}$ the *m-th component* of \mathcal{S} . Similarly, for a fixed element $n \in \overline{\mathbb{N}}$ we call the set $L_n := \mathbb{N} \times \{n\}$ the *n-th level*. So (m, n) is the unique element in the *m-th component* on level *n*. We call the elements on the ∞ -th level *limit points*, and refer to all the others as *finite elements* (although they are not “finite” in the sense of domain theory).

The order on \mathcal{S} is given by the product order plus the stipulation that the limit point (m, ∞) of the *m-th component* be above every element of levels 1 to *m*. It will be convenient later to write this formally as follows:

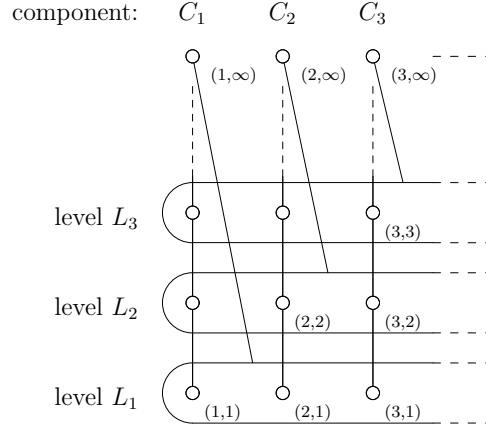
- $(m, n) \leq (m, n')$ if $n \leq n'$ ($m \in \mathbb{N}$, $n, n' \in \overline{\mathbb{N}}$)
- $(*, n) < (m, \infty)$ if $n \leq m$ ($*, n, m \in \mathbb{N}$)

As the Hasse diagram makes clear, the only non-trivial directed sets of \mathcal{S} are the chains contained in a component, with supremum the component's limit point. Since every finite element (m, n) is also below the limit points (m', ∞) , where $n \leq m'$, but not below any other finite element outside its own component, we see that none of them is *compact* in the sense of domain theory. It follows that \mathcal{S} is highly non-algebraic, and indeed it couldn't be algebraic as algebraic dcpos are always sober spaces in their Scott topology.

Let us take a closer look at the Scott topology on \mathcal{S} . The two defining conditions of Scott-closed sets manifest themselves in the following properties:

- Closed sets are lower sets: if the set contains a limit point (m, ∞) , then it must contain the component C_m and all levels L_1, \dots, L_m .
- Closed sets are closed under the formation of limits: if the set contains infinitely many elements of any one component then it must contain the limit point of that component.

Taken together, we obtain the following principle (which plays a crucial role in our construction as well):

Figure 1: Johnstone's non-sober dcpo \mathcal{S} .

(†) If a Scott-closed subset of \mathcal{S} contains infinitely many limit points, then it equals \mathcal{S} . This is because such a set contains infinitely many levels by the first property of Scott-closed sets, which means that it contains infinitely many elements of every component, and therefore contains the limit points of all components by the second property. Applying the first property again we see that it contains everything.

It is now easy to see that \mathcal{S} itself is an irreducible closed set: If we cover \mathcal{S} with two closed subsets then at least one of them must contain infinitely many limit points. By (†) that set then is already all of \mathcal{S} . Of course, \mathcal{S} is not the closure of a singleton since there is no largest element so we may conclude, as Peter Johnstone did in [7], that \mathcal{S} is not sober in its Scott topology. In the terminology of the last section, it also follows that \mathcal{S} is not strongly complete.

4. AN INFORMAL DESCRIPTION OF THE COUNTEREXAMPLE \mathcal{H}

4.1. The order sobrification of Johnstone's dcpo. The order sobrification $\widehat{\mathcal{S}}$ of \mathcal{S} is depicted in Figure 2. We will prove in Subsection 5.3 that the picture is correct, that is, $\widehat{\mathcal{S}}$ itself is indeed the only closed irreducible subset of \mathcal{S} which is not a principal downset.

For the moment, however, let us check whether \mathcal{S} is already a counterexample to the Ho-Zhao conjecture. Unfortunately, this is not the case, the Scott topologies of \mathcal{S} and $\widehat{\mathcal{S}}$ are not isomorphic. We see this as follows: The extra point at the top of $\widehat{\mathcal{S}}$ cannot be reached by a directed set; it is *compact* in the sense of domain theory. Also, the image of \mathcal{S} under $\eta_{\mathcal{S}}$ forms a Scott-closed subset B of $\widehat{\mathcal{S}}$. Together this means that the largest element of $\Gamma(\widehat{\mathcal{S}})$ is compact, while the largest element of $\Gamma(\mathcal{S})$ is not: It is the directed limit of the closed sets $\downarrow\{(m, \infty) \mid m \leq n\}$, $n \in \mathbb{N}$.

As an aside, we may also observe that the order sobrification of $\widehat{\mathcal{S}}$ would add yet another point, placed between the elements of B and the top element $\widehat{\mathcal{S}}$ which shows that order sobrification is not an idempotent process. In fact, our construction of \mathcal{H} addresses exactly this point, by making sure that the new elements that appear in the order sobrification are not compact and do not lead to new Scott-closed subsets.

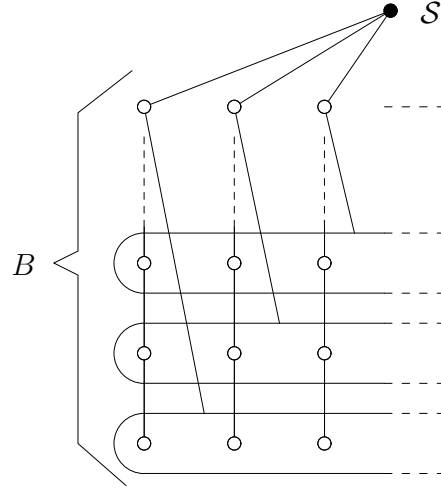


Figure 2: The order sobrification $\widehat{\mathcal{S}}$ of Johnstone's non-sober dcpo.

4.2. The counterexample \mathcal{H} . The construction of \mathcal{H} may be viewed as an infinite process. We begin with \mathcal{S} and for every finite level L_n of \mathcal{S} , we add another copy \mathcal{S}_n , whose infinite elements are identified with the elements of L_n . No order relation between the *finite* elements of two different $\mathcal{S}_n, \mathcal{S}_{n'}$ is introduced. Now the process is repeated with *each* finite level of *each* \mathcal{S}_n , adding a further $\mathbb{N} \times \mathbb{N}$ many copies of \mathcal{S} . We keep going like this *ad infinitum* and in this way ensure all elements are limit elements.

To make this a bit more precise, let \mathbb{N}^* be the set of lists of natural numbers. (Instead of “list” we also use the word “string”.) We denote lists by $n_1 n_2 \dots n_k$ and the empty list by ε . The result of adding the element n to the front of list s we write as ns , and for decomposing a list s into its first element and the rest we use the notation $\text{hd}(s)$ and $\text{tl}(s)$ (“head” and “tail”). Note that $\text{hd}(s) \in \mathbb{N}$ and $\text{tl}(s) \in \mathbb{N}^*$.

We use \mathbb{N}^* to index the many copies of Johnstone's example that make up our dcpo \mathcal{H} . In a first step, we let \mathcal{H}' be the disjoint union of all \mathcal{S}_s , $s \in \mathbb{N}^*$. We label individual elements of \mathcal{H}' with triples $(m, n, s) \in \mathbb{N} \times \overline{\mathbb{N}} \times \mathbb{N}^*$ in the obvious way. On \mathcal{H}' we consider the equivalence relation \sim which identifies the finite element (m, n, s) of \mathcal{S}_s with the infinite element (m, ∞, ns) of \mathcal{S}_{ns} , for all $m, n \in \mathbb{N}$ and $s \in \mathbb{N}^*$. We may now define \mathcal{H} as the quotient of \mathcal{H}' by \sim . We let the order on \mathcal{H} be the quotient order, that is, the smallest preorder such that the quotient map from \mathcal{H}' to \mathcal{H} is monotone. We will show in the next section that this is in fact an order and that \mathcal{H} is a dcpo. Figure 3 gives an impression of the resulting ordered set by showing the top of one component of \mathcal{H} .

The constituent copies of \mathcal{S} are still very much present in \mathcal{H} : For a fixed $s \in \mathbb{N}^*$ we may collect all elements of the form (m, n, s) , where $m \in \mathbb{N}$, $n \in \overline{\mathbb{N}}$, into a substructure of \mathcal{H} that is order-isomorphic to \mathcal{S} . We call it the *sheet* \mathcal{S}_s . On \mathcal{S}_s we have again the *finite levels* L_n^s consisting of the points $(1, n, s), (2, n, s), \dots$ for each $n \in \mathbb{N}$.

Glueing together the copies of \mathcal{S} in this way does not create any new directed sets other than those (essentially) contained in a single sheet \mathcal{S}_s . From this it follows that the characterisation of Scott-closed subsets given in Section 3 above is still valid. The principle (\dagger) that we derived from this, however, now has greater reach: Since every finite level in any sheet is simultaneously the set of limit points of a subsequent sheet, it holds

sheet \mathcal{S}_ε in the order sobrification then we obtain the structure displayed in Figure 4. The set B of point closures (indicated as open circles in Figure 4) no longer forms a Scott-closed subset of $\widehat{\mathcal{H}}$. Because a closed set must be downward closed, \overline{B} contains all A_n , $n \in \mathbb{N}$, and because it must be closed under taking limits, it contains A_ε as well, which means it equals all of $\widehat{\mathcal{H}}$.

The same considerations hold on any sheet $\widehat{\mathcal{S}}_s$, $s \in \mathbb{N}^*$, and we see that the Scott topology of $\widehat{\mathcal{H}}$ is the same as the sobrification topology, which in turn is always isomorphic to the topology of the original space, in our case the Scott topology of \mathcal{H} . Thus we have shown:

Theorem 4.1. The Scott topologies of \mathcal{H} and $\widehat{\mathcal{H}}$ are isomorphic.

On the other hand, \mathcal{H} and $\widehat{\mathcal{H}}$ are clearly not isomorphic as ordered sets: the latter has a largest element whereas the former does not. Thus we have:

Corollary 4.2. The category **DCPO** is not Γ -faithful.

5. FORMAL ARGUMENTS REGARDING \mathcal{H}

The structure of \mathcal{H} is sufficiently intricate to warrant a more formal examination of the claims made in the previous section.

5.1. The structure of \mathcal{H} as an ordered set. As explained informally above, we begin with the structure \mathcal{H}' , the disjoint union of infinitely many copies \mathcal{S}_s of Johnstone's dcpo, indexed by the set \mathbb{N}^* of strings of natural numbers. On \mathcal{H}' we define the equivalence relation \sim generated by

$$(m, \infty, s) \sim (m, \text{hd}(s), \text{tl}(s))$$

for all $m \in \mathbb{N}$, $s \in \mathbb{N}^* \setminus \{\varepsilon\}$.³ We note that each equivalence class with respect to \sim contains precisely the two elements (m, ∞, s) and $(m, \text{hd}(s), \text{tl}(s))$ which appear in the definition, except for the elements of the form (m, ∞, ε) (the maximal elements on sheet \mathcal{S}_ε) which are each in a class by themselves. For the arguments that follow below it is important to remember that \sim connects elements from *different* sheets, to wit, \mathcal{S}_s and $\mathcal{S}_{\text{tl}(s)}$.

The quotient preorder \lesssim is defined as the transitive closure of $< \cup \sim$. Any preorder gives rise to an equivalence relation but in our case this is just \sim again:

Proposition 5.1. The equivalence relation $\lesssim \cap \gtrsim$ derived from the quotient preorder is equal to \sim .

Proof. By definition, two elements (m, n, s) and (m', n', s') are related by \lesssim if and only if there is a sequence of elements (m_i, n_i, s_i) , $i = 1, \dots, k$ such that $(m, n, s) = (m_1, n_1, s_1) \sim (m_2, n_2, s_2) < (m_3, n_3, s_3) \sim \dots < (m_{k-1}, n_{k-1}, s_{k-1}) \sim (m_k, n_k, s_k) = (m', n', s')$. We consider the first two steps of such a sequence; since an element of the form $(*, \infty, *)$ is maximal in the sheet \mathcal{S}_* , the beginning of the sequence has to look as follows:

$$(m, \infty, ns) \sim (m, n, s) < (m', n', s)$$

with either $m = m'$ and $n < n'$, or $n \leq m'$ and $n' = \infty$. In the first case the sequence can continue with $(m, \infty, n's)$ but then we are in sheet $\mathcal{S}_{n's}$ where the head element of the

³If we make the head element of s explicit, then we can write the definition also as $(m, \infty, ns) \sim (m, n, s)$.

index $n's$ is strictly larger than in ns . In the second case, we are already in a sheet where the index s is strictly shorter than the starting index ns . This means that two elements (m, n, s) and (m', n', s') are related by $\lesssim \cap \gtrsim$ only if they are related by \sim already. \square

We can now define \mathcal{H} as the quotient \mathcal{H}'/\sim together with the quotient order \lesssim/\sim . We denote the quotient map with q .

Proposition 5.2. Restricted to any sheet \mathcal{S}_s , the map q is an order-isomorphism.

Proof. This follows from the fact that the equivalence relation connects elements from different sheets only. \square

This observation justifies speaking of the “sheet \mathcal{S}_s ” irrespective of whether we are in \mathcal{H} or in \mathcal{H}' .

Proposition 5.3. \mathcal{H} is a dcpo.

Proof. Elements of the form (m, ∞, s) are maximal in the sheet \mathcal{S}_s , so in order to move from a finite element (m, n, s) to a strictly larger one there is no other way than finding a larger element within \mathcal{S}_s , or to pass through an infinite element (m', ∞, s) within \mathcal{S}_s and then to switch over to the element $(m', \text{hd}(s), \text{tl}(s))$ on sheet $\mathcal{S}_{\text{tl}(s)}$ (and to carry on from there). However, the strings in \mathbb{N}^* are of finite length, so the process of switching sheet can only be performed finitely often. It follows that any directed set of \mathcal{H} contains a cofinal subset entirely contained in a single sheet. Because sheets are isomorphic to Johnstone’s dcpo \mathcal{S} by the preceding proposition, the supremum of the directed set can be found there as well. \square

This proof also shows the following:

Proposition 5.4. The quotient map q is Scott-continuous.

5.2. The Scott topology on \mathcal{H} . We are ready to show the main technical tool for establishing that \mathcal{H} is indeed a counterexample to the Ho-Zhao conjecture:

Proposition 5.5. The Scott topology on \mathcal{H} equals the quotient topology derived from the Scott topology on \mathcal{H}' and the equivalence relation \sim .

Proof. Since q is Scott-continuous, we have that $q^{-1}(O)$ is Scott-open in \mathcal{H}' for any Scott-open set O of \mathcal{H} . It follows that O belongs to the quotient topology. Conversely, if O is a member of the quotient topology, then $q^{-1}(O)$ is Scott-open in \mathcal{H}' , which means that the set $q^{-1}(O) \cap \mathcal{S}_s$ is Scott-open for every $s \in \mathbb{N}^*$. Since we showed in the proof of Proposition 5.3 that directed sets of \mathcal{H} are essentially contained in a single sheet, it follows that O is inaccessible by directed suprema; in other words, Scott-open since O is upper. \square

Below we will have to make several (and sometimes quite subtle) constructions of open (and closed) sets on \mathcal{H} . By the proposition above we are allowed to do this by exhibiting a Scott-open set O_s for each sheet \mathcal{S}_s such that $(m, \infty, s) \in O_s$ if and only if $(m, \text{hd}(s), \text{tl}(s)) \in O_{\text{tl}(s)}$. The compatibility with \sim together with the principle (\dagger) identified for Johnstone’s dcpo leads to the following principle for \mathcal{H} :

(\dagger) If a Scott-closed subset of \mathcal{H} contains infinitely many elements from any level L_n on any sheet \mathcal{S}_s , then it contains all of L_n .

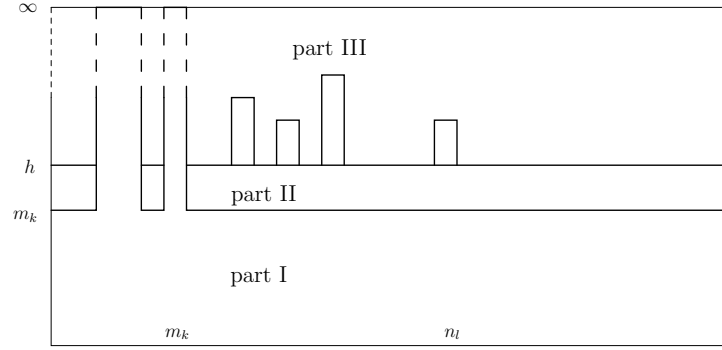


Figure 5: The three parts of a generic locally closed set.

This means that the closed (resp., open) sets $q^{-1}(F) \cap \mathcal{S}_s$ (where F is a closed set of \mathcal{H}) are *more specialized* than just being Scott-closed (respectively, Scott-open). Closed sets of the form $q^{-1}(F) \cap \mathcal{S}_s$ are called *locally closed* sets (respectively, *locally open*). We will need to know exactly what such sets look like and the next proposition spells this out. The characterisation follows from the definition of Scott-closedness, the order on \mathcal{S} , and (§). See Figure 5 for an illustration.

Proposition 5.6. A locally closed set $A \subseteq \mathcal{S}$ is either all of \mathcal{S} or can be uniquely written as the union of the following three parts:

- I:* $\bigcup_{i=1}^k \downarrow(m_i, \infty)$ where $m_1 < m_2 < \dots < m_k$ (the infinite columns)
- II:* $\bigcup_{1 \leq p \leq h} L_p$ where $h \geq m_k$ (the lower rectangle)
- III:* $\bigcup_{j=1}^l \downarrow(n_j, q_j)$ where $n_1 < n_2 < \dots < n_l$
and $n_j > h$ for $j = 1, \dots, l$ (the finite columns)

If any of the numbers k , h and l is equal to zero, then the corresponding part is empty, but because of the way the order is defined on \mathcal{S} , every component $\downarrow(m_i, \infty)$ from part I forces the presence of a lower rectangle of at least height m_i . The purpose of part II is to account for the possibility that A_s contains more complete rows than is required by the presence of limit points.

5.3. The irreducible subsets of Johnstone's \mathcal{S} . We already know that \mathcal{S} is not sober in its Scott topology since \mathcal{S} is closed and irreducible but not a point closure but we would like to establish that this is the only such subset. We do this in a fashion that prepares us for the more complicated case of \mathcal{H} .

Proposition 5.7. The closed irreducible subsets of \mathcal{S} are \mathcal{S} itself and the closures of singleton sets.

Proof. Assume that A is a closed proper subset of \mathcal{S} . We distinguish two cases:

Case 1: A contains no limit points. Let $(m, n) \in \mathbb{N} \times \overline{\mathbb{N}}$ be a maximal element of A . We claim that $B = A \setminus \downarrow(m, n)$ is Scott-closed. Indeed, since A contains no infinite chains at all we don't need to worry about closure under limits. Downward closure also holds because there is no order relationship between finite elements from different components

of \mathcal{S} . We therefore have the decomposition $A = B \cup \downarrow(m, n)$ which shows that A can only be irreducible if $B \subseteq \downarrow(m, n)$ and $A = \downarrow(m, n) = \text{cl}(m, n)$.

Case 2: A contains limit points. By the principle (\dagger) , there are only finitely many limit points in A , so let (m, ∞) be the one with the highest component number m . For a decomposition we set $D = \{(m, m+1), (m, m+2), \dots, (m, \infty)\}$ and $B = A \setminus D$, and again we have $A = B \cup \downarrow(m, \infty)$. If we can show that B is closed then the rest of the argument is as in the previous case. Closure under directed sups is clear because the only non-trivial chains of \mathcal{S} are those infinite subsets that are contained in a single component, and we have left all components of A unchanged except for C_m but there we have removed a cofinal part. Downward closure of B follows because $\uparrow(m, m+1) \setminus D$ consists only of the limit points $(m+1, \infty), (m+2, \infty), \dots$ and none of these were members of A to start with. \square

Proposition 5.8. Johnstone's dcpo \mathcal{S} is dominated.

Proof. The only nontrivial directed sets in $\widehat{\mathcal{S}}$ are the chains $(\downarrow(m, n))_{n \in \mathbb{N}}$ and their supremum is $\downarrow(m, \infty)$. If a closed irreducible subset A of $\widehat{\mathcal{S}}$ contains such a chain it must also contain its supremum, and because the supremum is a principal downset, it belongs to ∇A . \square

Remark 5.9. By Corollary 2.17, $\widehat{\mathcal{S}}$ is dominated. From the preceding result, \mathcal{S} is dominated. Hence $\Gamma(\mathcal{S}) \not\cong \Gamma(\widehat{\mathcal{S}})$; otherwise, because both \mathcal{S} and $\widehat{\mathcal{S}}$ are dominated, $\Gamma(\mathcal{S}) \cong \Gamma(\widehat{\mathcal{S}})$ would imply $\mathcal{S} \cong \widehat{\mathcal{S}}$, which we know is false. This argument then provides an alternate explanation as to why the pair of dcpos \mathcal{S} and $\widehat{\mathcal{S}}$ is not a counterexample to the Ho-Zhao conjecture.

5.4. The irreducible subsets of \mathcal{H} . Let us now apply a similar technique in order to identify precisely all irreducible subsets of \mathcal{H} . Our aim is to show the following:

Proposition 5.10. The closed irreducible subsets of \mathcal{H} are of the following two forms:

- (1) downsets of individual elements of \mathcal{H} ;
- (2) downsets of levels $L_n^s = \{(1, n, s), (2, n, s), (3, n, s), \dots\}$, where $s \in \mathbb{N}^*$, $n \in \mathbb{N}$, or $s = \varepsilon, n = \infty$.

Note that we could have expressed the second form as “downsets of levels L_∞^s ” without the need to treat $\mathcal{H} = \downarrow L_\infty^\varepsilon$ as a special case but for the arguments that follow we have found it advantageous to think of the level as consisting of finite elements.

The proof of the proposition is in three parts.

Proposition 5.11. The sets mentioned in the previous proposition are closed and irreducible.

Proof. Sets of the first kind can also be seen as closures of singleton subsets and such sets are always irreducible. Sets of the second kind are clearly irreducible by principle (\dagger) . The only concern is whether they are closed. For this we have to show that a set $A = \downarrow L_n^s$ is locally closed on each sheet \mathcal{S}_t . We use the characterisation provided in Proposition 5.6:

If $t = s$ then $A_t = A \cap \mathcal{S}_t$ consists of part II only, namely, the levels $L_1^s, L_2^s, \dots, L_n^s$. In case $s = \varepsilon, n = \infty$, A_t is all of \mathcal{S}_t .

If $t = ks$ for some $k \in \mathbb{N}$, then the limit points of \mathcal{S}_{ks} are identified with the level L_k^s of \mathcal{S}_s by the equivalence relation \sim . Therefore A_t is either empty (if $k > n$) or all of \mathcal{S}_{ks} (if $k \leq n$). In either case it is locally closed.

This argument repeats itself if ks is extended with further numbers. In all other cases (i.e., if s is not a suffix of t), A_t is empty. \square

Proposition 5.12. If a closed irreducible subset of \mathcal{H} contains a maximal element (m, n, s) but not all of the level L_n^s , then it equals the downset of (m, n, s) .

Proof. Assume that $A \subset \mathcal{H}$ contains (m, n, s) as a maximal element but not every element on the corresponding level L_n^s . We may assume that s is of minimal length and that on \mathcal{S}_s , n is maximal with these properties (i.e., if $n' > n$ then the level $L_{n'}^s$ contains no element of A). As in the proof of Proposition 5.7, we need to distinguish two cases:

Case 1: $n \neq \infty$: First observe that since n is maximal in A_s , part I of A_s is empty. We claim that there is a closed subset B which does not contain (m, n, s) such that $A = B \cup \downarrow(m, n, s)$. As in the proof of 5.7 we can then invoke irreducibility to conclude that $B \subset \downarrow(m, n, s)$ and hence $A = \downarrow(m, n, s)$. We define B via its locally closed sets $B_t = B \cap \mathcal{S}_t$, $t \in \mathbb{N}^*$. Here we go:

If $t = s$ then we first observe that because (m, n, s) is maximal in A and $n \neq \infty$, (m, n, s) is one of the generators of part III of A_s . This means we can let $B_s = A_s \setminus \{(m, n, s)\}$ and still have a subset that conforms with the structure exhibited in Proposition 5.6.

If $t = ks$ for some $k \in \mathbb{N}$, then the limit points of \mathcal{S}_{ks} are identified with the level L_k^s of \mathcal{S}_s by the equivalence relation. Since we altered A_s only on level n , we can restrict attention to the case where $k = n$. The element (m, n, s) appears in \mathcal{S}_{ns} as (m, ∞, ns) and since we removed the former from A_s we must remove the latter from A_{ns} , where it was one of the generators of part I of A_{ns} . (Note that we have assumed that A does not contain the whole level L_n^s , so A_{ns} does not contain all limit points of \mathcal{S}_{ns} .) However, in order to define B_{ns} it is not sufficient to remove the limit point (m, ∞, ns) as the result would not be closed under directed suprema. However, we can remove cofinally many elements of component C_m^{ns} , making sure we are not interfering with part II of A_{ns} . The result is that the infinite column generated by (m, ∞, ns) has been replaced with a finite column generated by (m, c, ns) where $c \geq h$ with h the height of the “lower rectangle” in the language of Proposition 5.6.⁴

The construction continues in the same vein as we add more numbers to the front of the string ns . We always have to convert an infinite column into a finite one in case the generating limit point was removed in the parent sheet.

If s is not a suffix of t , then we don't need to manipulate A_t at all in order to obtain B_t .

In summary, we have constructed the individual sets B_t in such a way that they conform to the characterisation of locally closed set in Proposition 5.6, that they are compatible with the equivalence relation \sim , and that all elements in $A_t \setminus B_t$ are below (m, n, s) . Crucially, B_s does not contain the element (m, n, s) we started out with.

Case 2: $n = \infty$: In this case we can also assume that $s = \varepsilon$ because otherwise the maximal element (m, ∞, s) is equivalent to $(m, \text{hd}(s), \text{tl}(s))$ on sheet $\mathcal{S}_{\text{tl}(s)}$, contradicting our assumption that the length of s is minimal. Since we know that A does not contain the whole level L_∞^ε , it contains only a finite subset of it by principle (\dagger) and (m, ∞, ε) is a generator of part I of A_ε . As before, we convert the infinite column generated by (m, ∞, ε) to a finite column generated by some (m, c, ε) and continue with this process on subsequent sheets. \square

For the third and final step in our proof of 5.10 we turn our attention to closed sets that do not contain any “isolated” maximal elements, that is, sets that contain the full level L_n^s whenever they contain the element (m, n, s) for some $m \in \mathbb{N}$. Levels are indexed by a string (indicating the sheet) and a number, but since the equivalence relation identifies L_∞^s with

⁴The observant reader will notice that we are using the fact that $h \geq m_k$ here.

$L_{\text{hd}(s)}^{\text{tl}(s)}$ we only need strings of numbers to index them. We say that level L^t is below L^s if $L^t \subseteq \downarrow L^s$. This induces an order \sqsubseteq between strings of numbers generated by the following two rules:

- $ns \sqsubseteq s$ for all $n \in \mathbb{N}$, $s \in \mathbb{N}^*$
- $ns \sqsubseteq n's$ for all $n \leq n' \in \mathbb{N}$, $s \in \mathbb{N}^*$

Proposition 5.13. With respect to \sqsubseteq , \mathbb{N}^* is a tree with root ε , in other words, $\uparrow s$ is linearly ordered for all $s \in \mathbb{N}^*$.

Proof. It is easy to write out the chain above s and to check that it is upwards closed, so must be all of $\uparrow s$:

$$\begin{aligned} s &= n_1 n_2 \cdots n_k \sqsubseteq (n_1 + 1) n_2 \cdots n_k \sqsubseteq \dots \\ &\sqsubseteq n_2 \cdots n_k \sqsubseteq (n_2 + 1) \cdots n_k \sqsubseteq \dots \\ &\vdots \\ &\sqsubseteq n_k \sqsubseteq n_k + 1 \sqsubseteq \dots \\ &\sqsubseteq \varepsilon \end{aligned}$$

□

A Hasse diagram of $(\mathbb{N}^*, \sqsubseteq)$ would look exactly the same as the one shown in Figure 3 (though there we used a different labelling scheme).

Proposition 5.14. A closed irreducible subset A of \mathcal{H} can only contain one complete level among its maximal elements.

Proof. Let L^s be a maximal level contained in A . We claim that $A \setminus \downarrow L^s$ is Scott-closed. Closure under suprema of directed sets is immediate since we are subtracting a downset. The issue at hand is downward closure, so assume there is $x \in \downarrow L^s \cap \downarrow (A \setminus \downarrow L^s)$, i.e., there is $y \in L^t \subseteq A \setminus \downarrow L^s$ with $x \leq y$. The element x belongs to some level L^u and since $\uparrow u$ is linearly ordered with respect to \sqsubseteq , we must have $s \sqsubseteq t$ or $t \sqsubseteq s$. In the former case, $s = t$ because we assumed L^s to be maximal in A . In the latter case we have $L^t \subseteq \downarrow L^s$ by the definition of \sqsubseteq . In either case we get a contradiction to the assumption that $y \in L^t \subseteq A \setminus \downarrow L^s$.

We decompose A as $\downarrow L^s \cup (A \setminus \downarrow L^s)$ and irreducibility now implies that the second component must be empty. □

Having identified exactly which subsets of \mathcal{H} are closed irreducible with respect to the Scott topology, we can now deliver on our promise, made at the end of Section 2, of giving an example of a non-dominated dcpo.

Proposition 5.15. The dcpo \mathcal{H} is not dominated.

Proof. By the preceding proposition, $\mathcal{H} \in \widehat{\mathcal{H}}$ and for every $n \in \mathbb{N}$, $\downarrow L_n^\varepsilon \in \widehat{\mathcal{H}}$. Furthermore, for each $n \in \mathbb{N}$, $\downarrow L_n^\varepsilon \subseteq \downarrow (n, \infty, \varepsilon)$ and hence $\mathcal{F} := \{\downarrow L_n^\varepsilon \mid n \in \mathbb{N}\}$ is a directed family in $\nabla \mathcal{H}$. But $\bigvee \mathcal{F}$ equals \mathcal{H} and it is not the case that $\mathcal{H} \triangleleft \mathcal{H}$. Thus, \mathcal{H} fails to be dominated. □

Reassured by the above result, we now proceed to the next subsection to establish that the dcpos \mathcal{H} and $\widehat{\mathcal{H}}$ form the desired counterexample to the Ho-Zhao conjecture.

5.5. The order sobrification of \mathcal{H} . Recall that the unit η of the order sobrification monad maps x to $\downarrow x$ and is Scott-continuous. We will employ this in our proof of Theorem 4.1:

Proof. We have the map $\kappa: \Gamma(\mathcal{H}) \rightarrow \Gamma(\widehat{\mathcal{H}})$ described in the previous section; it maps a closed set C to $\widehat{C} = \{A \in \widehat{\mathcal{H}} \mid A \subseteq C\}$. Because $\eta_{\mathcal{H}}$ is Scott-continuous we have the map η^{-1} in the opposite direction. We show that they are inverses of each other. For the first calculation let C be a Scott-closed subset of \mathcal{H} .

$$\begin{aligned} x \in \eta^{-1}(\widehat{C}) &\iff \downarrow x \in \widehat{C} && \text{(definition of } \eta) \\ &\iff \downarrow x \subseteq C && \text{(definition of } \widehat{C}) \\ &\iff x \in C && (C \text{ is a lower set}) \end{aligned}$$

For the other composition, let \mathcal{C} be a Scott-closed subset of $\widehat{\mathcal{H}}$.

$$\begin{aligned} A \in \mathcal{C} &\xrightarrow{(*)} \forall x \in A. \downarrow x \in \mathcal{C} && (\forall x \in A. \downarrow x \subseteq A \text{ as } A \text{ is a lower set in } \mathcal{H}, \\ &&& \text{and } \mathcal{C} \text{ is a lower set in } \widehat{\mathcal{H}}) \\ &\iff \forall x \in A. \eta(x) \in \mathcal{C} && \text{(definition of } \eta) \\ &\iff A \subseteq \eta^{-1}(\mathcal{C}) \\ &\iff A \in \widehat{\eta^{-1}(\mathcal{C})} && \text{(definition of } \widehat{(-)}) \end{aligned}$$

Our proof will be complete if we can show the reverse of the first implication in the calculation above. For this we use our knowledge about the elements of $\widehat{\mathcal{H}}$, that is, the irreducible closed subsets of \mathcal{H} established in the previous subsection (Proposition 5.10). For irreducible subsets of the form $\downarrow x$ the reverse of $(*)$ is trivially true, so assume that A is the downset of some level L_n^s for $s \in \mathbb{N}^*$, $n \in \mathbb{N}$, and $\forall x \in \downarrow L_n^s. \downarrow x \in \mathcal{C}$. In particular we have $\downarrow(m, n, s) \in \mathcal{C}$ for the elements (m, n, s) of L_n^s . Because all elements of level m on sheet \mathcal{S}_{ns} are below $(m, n, s) \sim (m, \infty, ns)$, we have $\downarrow L_m^{ns} \subseteq A$ for all $m \in \mathbb{N}$, and hence $\downarrow L_m^{ns} \in \mathcal{C}$ as the latter is downward closed. Finally, $A = \downarrow L_n^s = \downarrow L_\infty^{ns} = \text{cl}(\bigcup_{m \in \mathbb{N}} \downarrow L_m^{ns})$ so $A \in \mathcal{C}$ follows as desired. The case $s = \varepsilon, n = \infty$ is similar. \square

FINAL REMARKS

We have shown that the class of dominated dcpos is Γ -faithful and also seen that it is quite encompassing (Corollary 2.17). One may wonder whether there are other natural classes of dcpos on which Γ is faithful or whether **domDCPO** is in some sense “maximal.” Strictly speaking, the answer to this question is no, since the singleton class $\mathcal{C} = \{\mathcal{H}\}$ is (trivially) Γ -faithful, yet — as we showed — not contained in **domDCPO**. What is needed, then, is a proper definition of “maximal” before any attempt can be made to establish its veracity.

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REFERENCES

- [1] A. Alghoussein and A. S. Ibrahim. A Counterexample to the Generalized Ho-Zhao Problem. *Journal of Mathematics Research*, 7(3):137–140, 2015.
- [2] S. Abramsky and A. Jung. Domain Theory, volume 3 of Handbook of Logic in Computer Science. Clarendon Press, Oxford, 1994.
- [3] D. Drake and W.J. Thron. On the representation of an abstract lattice as the family of closed subsets of a topological space. *Trans. Amer. Math. Soc.*, 120:57–71, 1965.
- [4] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, and D.S. Scott. *Continuous Lattices and Domains*. Number 93 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2003.
- [5] R.-E. Hoffmann. Continuous posets, prime spectra of completely distributive complete lattices, and Hausdorff compactifications. *Lecture Notes in Mathematics*, 871:159–208, 1981.
- [6] W. K. Ho and D. Zhao. Lattices of Scott-closed sets. *Comment. Math. Univ. Carolinae*, 50(2):297–314, 2009.
- [7] P. T. Johnstone. Scott is not always sober. *Lecture Notes in Mathematics*, 81:333–334, 1981.
- [8] J. D. Lawson. The duality of continuous posets. *Houston Journal of Mathematics*, 5:357–394, 1979.
- [9] S. Papert. Which distributive lattices are lattices of closed sets? *Proceedings of the Cambridge Philosophical Society*, 55:172–176, 1959. [MR 21:3354].
- [10] D. S. Scott. Continuous lattices. In E. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136. Springer Verlag, 1972.
- [11] W. J. Thron. Lattice-equivalence of topological spaces. *Duke Mathematical Journal*, 29:671–679, 1962.
- [12] D. Zhao and T. Fan. Dcpo-completion of posets. *Theor. Comput. Sci.*, 411(22-24):2167–2173, 2010.